HAMILTON JACOBI EQUATION

The Hamilton – Jacobi equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself can not be solved completely.

The Hamilton – Jacobi equation is also the only formulation of mechanics in which the motion of a particle can be represented as wave

The Hamilton equations of motion for canonical variable q_i and P_i are

$$\dot{P}_K = -\frac{\partial H}{\partial q_K} \quad \text{and} \quad \dot{q}_K = \frac{\partial H}{\partial P_K}$$

If we make a canonical transformation from the old set of variables to a new set of variables (Q_K, P_K). Then the new equation of motion are

$$\dot{P}_K = -\frac{\partial H'}{\partial Q_K} \quad \text{and} \quad \dot{Q}_K = \frac{\partial H'}{\partial P_K} \quad \text{—— (1)}$$

Now if we require the transformed Hamiltonian H' is identically zero i.e $H' = 0$ the eqn of motion (1)

$$\dot{P}_K = 0 \quad \text{on} \quad \dot{Q}_K = 0 \quad \text{—— (2)}$$

$$P_K = \text{constant} \quad \text{and} \quad Q_K = \text{constant}$$

Thus the new co-ordinates and moments are constant in time and they are cyclic

The new Hamiltonian H' is related to the old Hamiltonian H by the relation

$$H' = H + \frac{\partial F}{\partial t} \quad \text{—③}$$

$$H' = 0$$

$$H(q_K, P_K, t) + \frac{\partial F}{\partial t} = 0 \quad \text{—④}$$

where $q_K = (q_1, q_2 \text{--} q_n)$ $P_K = (P_1, P_2 \text{--} P_n) = \beta_K$

For convenience, we take the generating funⁿ F as a function of the old co-ordinate q_K, the new constant momenta P_K and time t i.e, $F_2(q_K, P_K, t)$

then $p_K = \frac{\partial F_2}{\partial q_K}$ $Q_K = \frac{\partial F_2}{\partial P_j}$ —⑤

Therefore

$$H\left(q_K, \frac{\partial F_2}{\partial q_K}, t\right) + \frac{\partial F_2}{\partial t} = 0 \quad \text{—⑥}$$

Let us see, Physical meaning of the generating function $F_2(q_K, P_K, t)$. The total time derivative of F_2 is

$$\frac{\partial F_2}{\partial t} = \sum_{K=1}^{n} \frac{\partial F_2}{\partial q_K} \dot{q}_K + \sum_{n=1}^{n} \frac{\partial F_2}{\partial P_K} \dot{P}_K + \frac{\partial F_2}{\partial t}$$

Here $\dot{P}_K = 0$, $\frac{\partial F_2}{\partial t} = -H$ and $\frac{\partial F_2}{\partial q_K} = p_K$

Therefore

$$\frac{dF_2}{dt} = \sum_{K=1}^{n} p_K \dot{q}_K - H = L \quad \text{—⑦}$$

$$F_2 = \int L\,dt = S \quad \text{—⑧}$$

where S is the action of the system known as the Hamilton's principal function

writing $F_2 = S$

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \text{--- (9)}$$

This is known as Hamiltons Jacobi eqn.