

HAMILTON JACOBI EQUATION

The Hamilton - Jacobi equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself can not be solved completely.

The Hamilton - Jacobi equation is also the only formulation of mechanics in which the motion of a particle can be represented as wave

The Hamilton equations of motion for canonical variable q_i and p_i are

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{and} \quad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

If we make a canonical transformation from the old set of variables to a new set of variables (Q_k, P_k) . Then the new equations of motion are

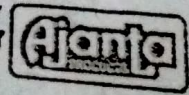
$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad \text{and} \quad \dot{Q}_k = \frac{\partial H'}{\partial P_k} \quad \text{--- (1)}$$

Now if we require the transformed Hamiltonian H' is identically zero i.e. $H' = 0$ the eqn of motion (1)

$$\dot{P}_k = 0 \quad \text{and} \quad \dot{Q}_k = 0 \quad \text{--- (2)}$$

$$P_k = \text{constant} \quad \text{and} \quad Q_k = \text{constant}$$

Thus the new co-ordinates and momenta are constant in time and they are cyclic



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The new Hamiltonian H' is related to the old Hamiltonian H by the relation

$$H' = H + \frac{\partial F}{\partial t} \quad \text{--- (3)}$$

$$H' = 0$$

$$H(q_k, p_k, t) + \frac{\partial F}{\partial t} = 0 \quad \text{--- (4)}$$

where, $q_k = (q_1, q_2 \dots q_n)$ $p_k = (p_1, p_2 \dots p_n) = P_k$

For convenience, we take the generating function F as a function of the old co-ordinates q_k , the new constant momenta p_k and time t i.e., $F_2(q_k, p_k, t)$

$$\text{then } p_k = \frac{\partial F_2}{\partial q_k} \quad q_k = \frac{\partial F_2}{\partial p_k} \quad \text{--- (5)}$$

Therefore

$$H\left(q_k, \frac{\partial F_2}{\partial q_k}, t\right) + \frac{\partial F_2}{\partial t} = 0 \quad \text{--- (6)}$$

Let us see, Physical meaning of the generating function $F_2(q_k, p_k, t)$. The total time derivative of F_2 is

$$\frac{dF_2}{dt} = \sum_{k=1}^n \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_{k=1}^n \frac{\partial F_2}{\partial p_k} \dot{p}_k + \frac{\partial F_2}{\partial t}$$

$$\text{Here } \dot{p}_k = 0, \quad \frac{\partial F_2}{\partial t} = -H \quad \text{and} \quad \frac{\partial F_2}{\partial q_k} = p_k$$

Therefore

$$\frac{dF_2}{dt} = \sum_{k=1}^n p_k \dot{q}_k - H = L \quad \text{--- (7)}$$

$F_2 = \int L dt = S$ --- (8)
where S is the action of the system known

as the Hamilton's principal function

writing $F_0 = S$

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \text{--- (9)}$$

This is known as Hamilton-Jacobi eqn.